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# SPECIAL BIRATIONAL TRANSFORMATIONS OF TYPE $(2, 1)$

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(JOINT WORK WITH JUN-MUK HWANG)

## 1. INTRODUCTION

This is a report on my joint work ([FH]) with Jun-Muk Hwang from KIAS. We are interested in the classification of *special birational transformations*: namely a birational transformation  $\Phi : \mathbb{P}^n \dashrightarrow Z \subset \mathbb{P}^N$  where  $Z \subset \mathbb{P}^N$  is a nonsingular projective variety of Picard number 1 such that the base locus  $S \subset \mathbb{P}^n$  of  $\Phi$  is irreducible and nonsingular. The map  $\Phi$  is called of *type*  $(a, b)$  if the rational map  $\Phi$  is given by a linear system belonging to  $\mathcal{O}_{\mathbb{P}^n}(a)$  and the inverse rational map  $\Phi^{-1}$  is given by a linear system belonging to  $\mathcal{O}_Z(b)$ .

When  $Z = \mathbb{P}^n$ , such a transformation is called a *special Cremona transformation*. In [ES], Ein and Shepherd-Barron have classified special Cremona transformations of type  $(2, 2)$ , by proving that the base locus  $S$  is one of the four Severi varieties classified by Zak ([Z]). Recently, special Cremona transformations on  $\mathbb{P}^n$  of type  $(2, 3)$  or of type  $(2, 5)$  or when  $n$  is odd have been classified by Russo ([R]).

Let  $\Phi : \mathbb{P}^n \dashrightarrow Z$  be a special Cremona transformation of type  $(2, 2)$ . If we restrict  $\Phi$  to a general hyperplane section, we obtain a special birational transformation of type  $(2, 2)$   $\Phi' : \mathbb{P}^{n-1} \dashrightarrow \mathbb{Q}^{n-1}$  with base locus being a hyperplane section of a Severi variety. Conversely, every special birational transformation of type  $(2, 2)$  from  $\mathbb{P}^n$  to  $\mathbb{Q}^n$  is of this form, as shown by Staglianò ([St]).

Motivated by the paper of [AS2] (and by the lack of examples), we have the following

**Speculation 1.1.** Let  $\Phi : \mathbb{P}^n \dashrightarrow Z$  be a special birational transformation of type  $(2, b)$ . If  $Z$  is different to  $\mathbb{P}^n$  and  $\mathbb{Q}^n$ , then  $b \leq 2$ . Furthermore if  $b = 2$ , then  $n$  is even and  $Z \simeq \text{Gr}(2, n/2 + 2)$ .

Here are some supporting results for this speculation: 1) If  $Z$  is a linear section of some rational homogeneous manifolds, then Speculation 1.1 holds by [AS2] (Corollary 3.21). 2) Speculation 1.1 with  $b = 2$  holds if  $Z$  is a *LQEL* manifold by [AS2] (Theorem 3.6). 3) Speculation 1.1 with  $b = 2$  holds if  $Z$  is a hypersurface by [Li].

In our paper ([FH]), we have obtained a complete classification for special birational transformations of type  $(2, 1)$ , hence confirming Speculation 1.1 for the case when  $b = 1$ . This classification can be described in terms of the classification of the base locus  $S \subset \mathbb{P}^n$ , which is contained in a hyperplane  $\mathbb{P}^{n-1} \subset \mathbb{P}^n$  for the type  $(2, 1)$ . Our main result is the following.

**Theorem 1.2** ([FH]). *The base locus  $S^d \subset \mathbb{P}^{n-1}$  of a special birational transformations of type  $(2, 1)$  is projectively equivalent to one of the following:*

- (a)  $\mathbb{Q}^d \subset \mathbb{P}^{d+1}$  for  $d \geq 1$ ;
- (b)  $\mathbb{P}^1 \times \mathbb{P}^{d-1} \subset \mathbb{P}^{2d-1}$  for  $d \geq 3$ ;
- (c) the 6-dimensional Grassmannian  $\text{Gr}(2, 5) \subset \mathbb{P}^9$ ;
- (d) the 10-dimensional Spinor variety  $\mathbb{S}_5 \subset \mathbb{P}^{15}$ ;
- (e) a nonsingular codimension  $\leq 2$  linear section of  $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ ;
- (f) a nonsingular codimension  $\leq 3$  linear section of  $\text{Gr}(2, 5) \subset \mathbb{P}^9$ .

The corresponding  $Z$  is given by

- (a')  $\mathbb{Q}^{d+2}$  for  $d \geq 1$ ;
- (b')  $\text{Gr}(2, d+2)$  for  $d \geq 3$ ;
- (c') the 10-dimensional Spinor variety  $\mathbb{S}_5$ ;
- (d') the 16-dimensional  $E_6$ -variety  $\mathbb{OP}^2$ ;
- (e') a nonsingular codimension  $\leq 2$  linear section of  $\text{Gr}(2, 5)$ ;
- (f') a  $\mathbb{P}^4$ -general linear section of  $\mathbb{S}_5 \subset \mathbb{P}^{15}$  of codimension  $\leq 3$ .

The birational map  $\Phi$  is given by  $H^0(\mathbb{P}^n, \mathcal{I}_S(2))$  except in case (b) with  $d \geq 6$ , where it can be given by some subspaces of  $H^0(\mathbb{P}^n, \mathcal{I}_S(2))$ .

By [IP] (Remark 3.3.2), all non-singular linear sections of  $\text{Gr}(2, 5)$  of a fixed codimension  $\leq 3$  are projectively equivalent. All non-singular hyperplane sections of  $\mathbb{S}_5$  are projectively equivalent. We can show that smooth codimension 2 linear sections of  $\mathbb{S}_5$  are isomorphic but smooth codimension 3 linear sections of  $\mathbb{S}_5$  have moduli. Thus in the case (f'), these linear sections of codimension 3 are not general, but only general among those containing a  $\mathbb{P}^4$ .

Consider the case (b) with  $d \geq 6$ . The corresponding birational transformation is  $\phi^\circ : \mathbb{P}(V) \simeq \mathbb{P}^{2d} \dashrightarrow Z := \text{Gr}(2, d+2) \subset \mathbb{P}(V \oplus W)$  with  $Y = \text{Gr}(2, d) \subset \mathbb{P}W = \mathbb{P}(\wedge^2 \mathbb{C}^d)$ . Note that  $Z \cap \mathbb{P}W = Y$  and  $\text{Sec}(Z) \cap \mathbb{P}W = \text{Sec}(Y)$ . Take any linear subspace  $L \subset W$  such that  $\mathbb{P}L \cap \text{Sec}(Y) = \emptyset$ , then  $\mathbb{P}L \cap \text{Sec}(Z) = \emptyset$ . Let  $p_L : \mathbb{P}(V \oplus W) \dashrightarrow \mathbb{P}(V \oplus W/L)$  be the projection from  $\mathbb{P}L$ . Then  $p_L$  sends isomorphically  $Z$  to  $Z_L \subset \mathbb{P}(V \oplus W/L)$  and  $Y$  to  $Y_L \subset \mathbb{P}(W/L)$ . The map  $\phi_L^\circ := p_L \circ \phi^\circ : \mathbb{P}(V) \dashrightarrow Z_L$  is a special birational transformation of type  $(2, 1)$  with  $Y(\sigma_L) = Y_L \simeq Y \subset \mathbb{P}(W/L)$ .

## 2. OUTLINE OF THE PROOF

**2.1. Basic properties of special birational transformation of type (2, 1).** For a projective subvariety  $S \subset \mathbb{P}^n$ , its secant variety  $\text{Sec}(S)$  is the closure of the loci of lines which intersect  $S$  at two or more points. The secant defective is given by  $\delta = 2 \dim(S) + 1 - \dim \text{Sec}(S)$ . For  $u \in \text{Sec}(S) \setminus S$ , the entry locus  $\Sigma_u$  is the closure of the locus on  $S$  of secant lines through  $u$ . We call  $S \subset \mathbb{P}^n$  is a *QEL*-manifold if it is smooth and  $\Sigma_u$  is a smooth hyperquadric in its linear span  $C_u := \langle \Sigma_u \rangle$ .

From now on, let  $\Phi : S^d \subset \mathbb{P}^n \dashrightarrow Z$  be a special birational transformation of type (2, 1). As easily seen, for a line  $\ell \subset \mathbb{P}^n$  not contained in  $S$ , the image  $\Phi(\ell)$  is a point (resp. a line, a conic) if  $\ell \cdot C$  is two points (resp. one point, empty).

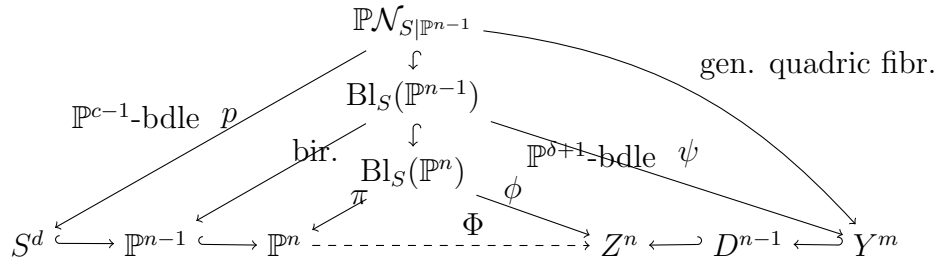
**Lemma 2.1** ([ES]).  *$S \subset \text{Sec}(S) = \mathbb{P}^{n-1}$  is a QEL-manifold. As a consequence, the secant defective of  $S$  is  $\delta = 2d + 2 - n$ .*

From this, we deduce that for  $u \in \mathbb{P}^{n-1}$  general,  $C_u \simeq \mathbb{P}^{\delta+1}$  is contracted to one point by  $\Phi$ . The following implies that for any  $u \in \mathbb{P}^{n-1} \setminus S$ , we always have  $C_u \simeq \mathbb{P}^{\delta+1}$ . This property is called *strong QEL* in [AS2].

**Lemma 2.2.** *The subvariety  $Y := \Phi(\mathbb{P}^{n-1}) \subset Z$  is smooth.*

The poof uses in a crucial way the smoothness of  $Z$ . Take any point  $y \in \mathbb{P}^n \setminus \text{Sec}(S)$ , we consider the  $\mathbb{C}^*$ -action on  $\mathbb{P}^n$  with orbits being lines through  $y$ . It fixes the point  $y$  and the hyperplane  $\mathbb{P}^{n-1} = \text{Sec}(S)$ . This action induces a  $\mathbb{C}^*$ -action on  $Z$ , with  $Y$  being an irreducible component of the fixed locus, hence  $Y$  is smooth since  $Z$  is smooth.

By [ES] (Theorem 1.1),  $\text{Bl}_S(\mathbb{P}^n)$  is also the blow-up of  $Z$  along  $Y$ , hence the map  $\psi : \text{Bl}_S(\mathbb{P}^{n-1}) \rightarrow Y$  is a  $\mathbb{P}^{\delta+1}$ -bundle. Let  $c := n - 1 - d$  be the codimension of  $S$  in  $\mathbb{P}^{n-1}$ . The notations and basic properties of our situation are summarised in the following diagram.



An easy argument using previous diagram shows that

**Proposition 2.3.** *The Euler numbers of  $S$  and  $Y$  are related by*

$$\chi(S) = \frac{(\delta + 2)\chi(Y) - n}{n - d - 2}.$$

**2.2. Classification of  $Y$ .** A key observation is that the fibers of  $p : \mathbb{P}\mathcal{N}_{S|\mathbb{P}^{n-1}} \rightarrow S$  are mapped to linear subspaces in  $Y$  and such  $\mathbb{P}^{c-1}$ 's cover  $Y$ . On the other hand, we have  $\dim Y = n - 1 - (\delta + 1) = 2(c - 1)$  by Lemma 2.1. By [S] and note that  $Y$  has Picard number 1, we obtain

**Proposition 2.4.** *The projective subvariety  $Y \subset \mathbb{P}^{a-1}$  is isomorphic to one of the following:*

- (Y1)  $\mathbb{P}^{2(c-1)} \cong Y = \mathbb{P}^{a-1}$ ;
- (Y2) a nonsingular quadric hypersurface  $\mathbb{Q}^{2(c-1)} \cong Y \subset \mathbb{P}^{a-1} \cong \mathbb{P}^{2c-1}$ ; or
- (Y3) a biregular projection of the Plücker embedding  $\mathrm{Gr}(2, \mathbb{C}^{c+1}) \subset \mathbb{P}(\wedge^2 \mathbb{C}^{c+1})$ .

We have the following topological consequence.

**Corollary 2.5.** *All odd Betti numbers of  $S$  vanish. In particular, the Euler number of  $S$  satisfies  $\chi(S) \geq d + 1$ .*

**2.3. Classification when  $Y = \mathbb{P}^{2(c-1)}$ .**

**Proposition 2.6.** *Assume that  $Y = \mathbb{P}^{2(c-1)}$ , then  $S \subset \mathbb{P}^{n-1}$  is projectively equivalent to one of the following:*

- (i)  $\mathbb{Q}^d \subset \mathbb{P}^{d+1}$ ;
- (ii)  $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ ;
- (iii)  $\mathrm{Gr}(2, 5) \subset \mathbb{P}^9$ ;
- (iv) a general hyperplane section of (ii);
- (v) a general codimension  $\leq 2$  linear section of (iii);
- (vi) a general codimension-3 linear section of (iii);
- (vii) a general codimension-2 linear section of (ii).

Here is the idea for a proof different to that presented in [FH]. If  $c = 1$ , then  $S \subset \mathbb{P}^{n-1}$  is a hyperquadric and we are done. Assume now  $c \geq 2$ . Let  $g(S)$  be the sectional genus of  $S$ . In this case, we have  $a - 1 = 2(c - 1)$ , hence by Theorem 1 [AS1], we have either  $\deg(S) = 2, c = 2, g(S) = 0$  or  $\deg(S) = 5, c = 3, g(S) = 1$ . Now the Proposition follows from the classification of low degree varieties or from the results on QEL-manifolds obtained by Russo in [R].

**2.4. Classification when  $Y = \mathbb{Q}^{2(c-1)}$ .** The following result is an easy consequence of Proposition 2.3.

**Proposition 2.7.** *For a special quadratic manifold  $S^d \subset \mathbb{P}^{n-1}$  with codimension  $c$  and  $n \geq 3$ , assume that  $Y = \mathbb{Q}^{2(c-1)}$ . Then either  $\delta \geq \frac{d}{2}$  or  $\delta = \frac{d}{2} - 1$ .*

**Proposition 2.8.** *For a special quadratic manifold  $S \subset \mathbb{P}^{n-1}$  of codimension  $c$ , assume that  $Y = \mathbb{Q}^{2(c-1)}$ . Then  $S \subset \mathbb{P}^{n-1}$  is projectively equivalent to one of the following*

- (i) *the Segre 4-fold  $\mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$ ;*
- (ii) *the 10-dimensional Spinor variety  $\mathbb{S}_5 \subset \mathbb{P}^{15}$ .*

By [R],  $d$ -dimensional QEL-manifolds with  $\delta \geq d/2$  have been completely classified. Hence by Proposition 2.7, the remaining case is  $\delta = \frac{d}{2} - 1$ . If  $\delta \leq 2$ , then we get  $d \leq 6$  and we do a case-by-case check. Now assume  $\delta \geq 3$ . By the divisibility theorem of [R],  $d - \delta = \delta + 2$  is divisible by  $2^{\lfloor \frac{\delta-1}{2} \rfloor}$  which implies that we have only two possibilities:  $(\delta, d, n) = (4, 10, 18)$  or  $(\delta, d, n) = (6, 14, 24)$ . For the first case, we can show the VMRT of  $S$  is  $\mathbb{P}^1 \times \mathbb{P}^4 \subset \mathbb{P}^9$ , hence  $S$  is isomorphic to  $\text{Gr}(2, 7)$  by [Mo], which cannot be embedded linearly normally into  $\mathbb{P}^{17}$ .

The case  $(\delta, d, n) = (6, 14, 24)$  is difficult to be ruled out. The contradiction comes from a delicate structure on intersections of entry loci of  $S$ . At one hand, we show that every two general entry loci  $\Sigma_i, i = 1, 2$  of  $S$  through a general point  $x \in S$  will intersect along  $\mathbb{P}^1$ , hence the lines on  $\Sigma_i$  through  $x$  give two entry loci on the VMRT  $\mathcal{C}_x$  of  $S$  at  $x$ , intersecting at a single point (corresponding to the line  $\Sigma_1 \cap \Sigma_2$ ). On the other hand, we can show that  $\mathcal{C}_x$  is a codimension 2 linear section of the 10-dimensional spinor variety  $\mathbb{S}_5 \subset \mathbb{P}^{15}$ . Any two entry loci of such a linear section intersect along a  $\mathbb{P}^k$  with  $k \geq 1$  if non-empty, which is a contradiction.

**2.5. Classification when  $Y = \text{Gr}(2, c + 1)$ .** To handle the case (Y3) of Proposition 2.4, we need the following result.

**Proposition 2.9** (Theorem 2.17 [AS2]). *Let  $S' \subset \mathbb{P}^{2k}$  be a quadratic manifold. Assume that the rational map  $\psi^o : S' \dashrightarrow Y(S')$  is birational and  $Y(S')$  is biregular to  $\text{Gr}(2, k + 2)$ . Then  $S' \subset \mathbb{P}^{2k}$  is one of the rational normal scrolls. In particular,  $S'$  has Picard number 2 and is covered by lines.*

**Proposition 2.10.** *Let  $S \subset \mathbb{P}^{n-1}$  be a special quadratic manifold of codimension  $c \geq 4$  with  $Y(S) = \text{Gr}(2, c + 1)$ . Then  $S \subset \mathbb{P}^{n-1}$  is projectively equivalent to  $\mathbb{P}^1 \times \mathbb{P}^c \subset \mathbb{P}^{2c+1}$ .*

The idea of the proof is the following: we take a general codimension- $(\delta + 1)$  linear section  $S' \subset \mathbb{P}^{n-\delta-2}$  of  $S \subset \mathbb{P}^{n-1}$ . This gives a birational

map  $\psi^\circ : S' \subset \mathbb{P}^{n-\delta-2} \dashrightarrow Y(S') = \text{Gr}(2, c+1)$ , hence  $S'$  is a rational normal scroll by 2.9. Except a few lower dimensional cases, this implies that  $S$  has Picard number 2 with  $\delta > 0$ . But then  $S$  becomes a conic-connected manifold with Picard number 2, which have been completely classified by [IR], and then we are done.

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